PHYSICAL REVIEW E **88**, 022115 (2013)

Non-Gaussian propagator for elephant random walks

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(Received 21 January 2013; published 9 August 2013)

For almost a decade the consensus has held that the random walk propagator for the elephant random walk (ERW) model is a Gaussian. Here we present strong numerical evidence that the propagator is, in general, non-Gaussian and, in fact, non-Lévy. Motivated by this surprising finding, we seek a second, non-Gaussian solution to the associated Fokker-Planck equation. We prove mathematically, by calculating the skewness, that the ERW Fokker-Planck equation has a non-Gaussian propagator for the superdiffusive regime. Finally, we discuss some unusual aspects of the propagator in the context of higher order terms needed in the Fokker-Planck equation.

DOI: 10.1103/PhysRevE.88.022115 PACS number(s): 05.40.Fb, 05.70.Fh, 73.22.Gk, 89.75.−k

I. INTRODUCTION

Anomalous diffusion and transport [1–14] requires at least one of two ingredients to avoid the consequences of the central limit theorem: (i) long range power law correlations in time or (ii) diverging variance for the distribution of velocities or jump sizes. Classic examples of these two mechanisms are fractional Brownian motion and Lévy flights, respectively [15]. The relation to non-Gaussian propagators has also been discussed in the literature, as in fractional Fokker-Planck equations generating non-Gaussian, non-Lévy probability distributions [16–18] and in intermittent maps exhibiting anomalous diffusion [19–21]. Here we focus on temporal correlations, i.e., memory effects. Random walks with long range power law correlations cannot (usually) be reduced to Markovian processes. Non-Markovian processes are still not very well understood, but they are ubiquitous in physical, biological, and socioeconomic phenomena [10,13–15,22–26], so their importance is widely recognized. Very few non-Markovian random walk models are exactly solvable. Notable among them is the elephant random walk (ERW) model, in which the random walker has memory access to the complete history of the random walk. Not surprisingly, the ERW allows anomalous diffusion. Specifically, there is superdiffusion in which the mean squared displacement grows faster than linearly in time. For almost a decade it had been thought that the exact solution of the ERW model in both normal and anomalous diffusion regimes is given by a simple Gaussian random walk propagator. The term propagator refers to the fundamental solution or Green's function in the context of the probability density function for a random walk. Here we show that the behavior is, in fact, unexpectedly rich: we prove that the ERW propagator is non-Gaussian in the superdiffusive regime. This surprising result opens many further problems, a few of which we briefly discuss.

II. THE MODEL

The ERW was introduced by two of us in 2004 [13]. The ERW is a one-dimensional random walk in which the position X_{t+1} at time $t+1$ is given by a probabilistic recurrence relation:

$$X_{t+1} = X_t + \sigma_{t+1}. \tag{1}$$

The position at time t is thus $X_t = \sum_{t'=1}^{t} \sigma_{t'}$. Here $\sigma_{t+1} = \pm 1$ is a binary random sequence that contains two-point correlations (i.e., memory).

At time $t+1$ a previous time $1 \leqslant t' < t+1$ is randomly drawn from a uniform probability distribution. The current step direction σ_{t+1} is then randomly decided on the value as follows:

$$\sigma_{t+1} = \begin{cases} +\sigma_{t'} & \text{with probability } p, \\ -\sigma_{t'} & \text{with probability } 1-p, \end{cases} \tag{2}$$

so that for $p > 1/2$ there is positive feedback and for $p < 1/2$ there is negative feedback. The ERW with bias assumes that the first step always goes to the right, i.e., $\sigma_1 = +1$, so $X_1 = 1$ assuming $X_0 = 0$. This initial condition introduces an initial bias in the walk with obvious consequences to the model statistics. The ERW model presents four types of diffusion regimes, namely, normal diffusion for $p \leqslant 1/2$ ($\alpha \leqslant 0$), normal diffusion with escape for $1/2 < p < 3/4$ ($0 < \alpha < 1/2$), marginal diffusion for $p = 3/4$ ($\alpha = 1/2$), and superdiffusive diffusion for $p > 3/4$ ($\alpha > 1/2$).

The conditional probability $P(X_2,t_2|X_1,t_1)$ to find the walker at X_2 at time t_2 given a previous position X_1 at time t_1 is given by [Eq. (18) in [13]]

$$\begin{aligned} P(Y,t+1|X_0,0) &= \frac{1}{2}\left[1 - \frac{\alpha(Y+1)}{t}\right] P(Y+1,t|X_0,0) \\ &\quad + \frac{1}{2}\left[1 + \frac{\alpha(Y-1)}{t}\right] P(Y-1,t|X_0,0), \end{aligned} \tag{3}$$

where $\alpha = 2p - 1$. The usual continuum limit leads to the Fokker-Planck equation [Eq. (20) in [13]],

$$\frac{\partial}{\partial t} P(x,t) = \frac{1}{2}\frac{\partial^2}{\partial x^2} P(x,t) - \frac{\alpha}{t}\frac{\partial}{\partial x}[x P(x,t)], \tag{4}$$

where

$$P(X_t - X_0, t) \equiv P(X_t, t | X_0, 0). \tag{5}$$

Direct substitution shows that the solution for a Dirac δ initial condition, i.e., the random walk propagator, is a Gaussian with a time dependent diffusion constant.

III. NUMERICAL RESULTS

Having introduced the ERW, we next report numerical evidence (Fig. 1) that the propagator is not, in fact, Gaussian. In what follows, we use the following non-Gaussian distribution for comparison:

$$P^*(x,t) = [1 + g(t)(x - \overline{x})]P(x,t), \tag{6}$$

where $g(t)$ and $\overline{x} \equiv \overline{x}(t)$ are fitting parameters and $P(x,t)$ represents a Gaussian distribution [solution of Eq. (4)]. We will always refer to P^* as the modified Gaussian. This functional form of P^* will be justified in the discussion below.

The numerical evidence we present below is a comparison of the simulation results with P and with P^*. Fittings to data obtained by Monte Carlo simulations with $p = 0.8$, i.e., $\alpha = 0.6$, within the superdiffusive regime, are shown in Fig. 1. In this case the walk with the initial bias of the original ERW is considered, i.e., with a deterministic choice $\sigma_1 = +1$. Figure 1(a) shows the residuals calculated as the difference between the logarithm of the fit with respect to the numerical data (long-dashed black line). The fitted distributions are the Gaussian (red dots) and the modified Gaussian (blue crosses) distributions. Clearly, the modified Gaussian provides a better fit than the Gaussian. The inset shows the logarithm of the same distributions (semilog plot), so that Gaussians become parabolas. The Gaussian fit (red dots) provides the worst fit, while the modified Gaussian (blue crosses) fit is the best. A closer look reveals a skewness in this figure (a result of the initial bias). In fact we have shown by Monte Carlo simulations that the third central moment does not vanish in the asymptotic limit for $p = 0.8$.

In order to remove the effect of the initial bias, we ran identical simulations but with completely random (and equiprobable) stochastic initial conditions $\sigma_1 = \pm 1$. The distribution for this unbiased ERW is shown in Fig. 1(b). Even after removing the skewness, the propagator remains non-Gaussian for $p = 0.8$.

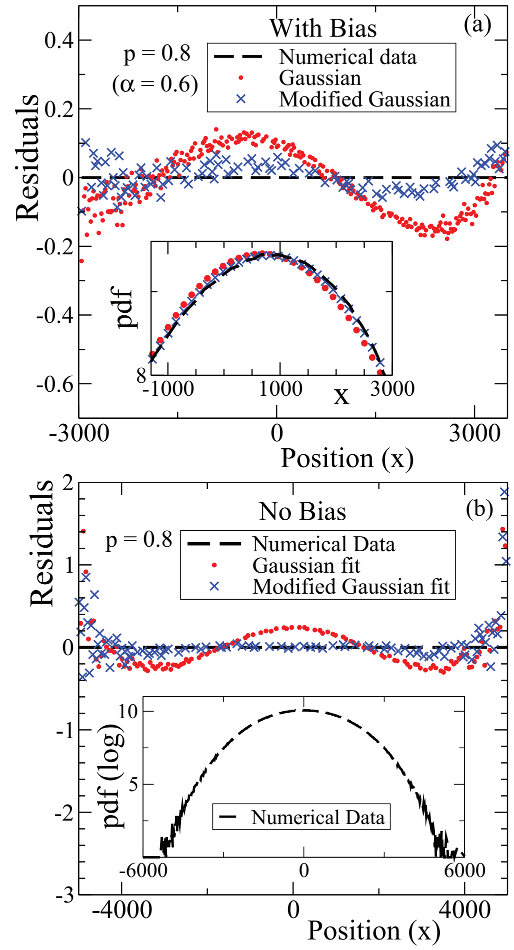

FIG. 1. (Color online) Numerically estimated ERW propagator in the superdiffusive regime for $\alpha = 0.6$ ($p = 0.8$). The red dots (blue crosses) refer to a Gaussian [modified Gaussian, Eq. (6)] fit to the numerically estimated pdf. The numerical data are shown as a long-dashed black curve. (a) shows the residuals of the fittings to the natural logarithm of the numerical data for the biased case. The inset shows the logarithm of the distributions focusing on the central part of the distributions to allow for a more accurate visual inspection. The modified Gaussian (crosses) provides the best fitting. (b) shows the nonbiased case. The residuals of the logarithm of the position distributions against the position (x) are shown in the main panel. It is clear that the Gaussian statistics (red dots) does not match the numerical data. The best fit is provided by the modified Gaussian (blue crosses). The numerical results are also shown in the inset as a log-linear plot, representing the logarithm of the distribution against the position (x) in the horizontal axis. With this choice of scale a Gaussian is represented by a parabola. The resemblance to a parabola, however, is misleading, as confirmed by the residuals (red dots) in the main panel. The numerical results were obtained for $T_{\max} = 50\,000$ and 4×10^7 runs and the histograms were drawn by counting exactly the number of occurrences of a given position. In this way we avoided the use of boxes (bins), which causes data smoothing and may hide the details by averaging all the points inside the boxes.

IV. ANALYTICAL RESULTS

Motivated by this strong numerical evidence of non-Gaussianness, we sought a deeper theoretical understanding. We next provide a mathematical proof that the propagator is non-Gaussian for $\alpha > 1/2$, corresponding to the superdiffusive regime $p > 3/4$. The strategy of our proof by contradiction is to exploit the fact that the relative skewness should vanish (at least in the long term limit) for Gaussian propagators.

A. Skewness

The exact moments will be represented by $x_n(t) \equiv \langle x_n(t) \rangle$ and can be determined from the exact recurrence equation (see [13]) given by

$$x_n(t+1) = h_n(t) + g_n(t) x_n(t), \tag{7}$$

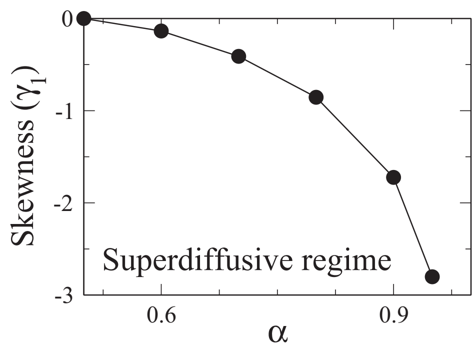

FIG. 2. Analytically calculated skewness γ_1 of the distribution for the biased ERW from Eqs. (A8), (A9), and (A10) for some selected values of α with $\alpha > 1/2$ within the superdiffusive regime. The nonzero skewness is proof of non-Gaussian behavior.

which is solved by

$$x_n(t) = x_n(1)\prod_{k=1}^{t-1} g_n(k) + \sum_{m=1}^{t-1}\left(h_n(m)\prod_{k=m+1}^{t-1} g_n(k)\right). \tag{8}$$

Here $h_n(t)$ and $g_n(t)$ are known functions that change for each moment. Expressions for $h_n(t)$ and $g_n(t)$ for $n = 1,2$ were given in the original paper [13]. Below we show how to get these functions in general.

We consider the third central moment $\Delta_3 = \langle (X_t - x_1(t))^3 \rangle$, the standard deviation $\sigma = \langle (X_t - x_1(t))^2 \rangle^{1/2}$, and the skewness

$$\gamma_1 = \Delta_3/\sigma^3 \tag{9}$$

and present analytical proof that the skewness γ_1 is nonzero in the superdiffusive regime, even in the long time limit. A recurrence relation for the position averaged moments can be written directly from the basic properties of the walker as (see Ref. [27] for details)

$$x_n(t+1) = \delta + \left(1 + \frac{n\alpha}{t}\right)x_n(t) + \sum_{l=1}^{s(n)}\left[\binom{n}{2l} + \binom{n}{2l+1}\frac{\alpha}{t}\right]x_{n-2l}(t), \tag{10}$$

where $\delta = [1 + (-1)^n]/2$, $s(n) = (n - \delta - 1)/2$. Recurrence relation (10) can be used in conjunction with Eq. (7) to get $h_n(t)$ and $g_n(t)$. The moments can then be obtained from Eq. (8), which allows us to determine Δ_3 using the conditions $x_1(1) = x_2(1) = x_3(1) = 1$. Keeping only the dominant terms, we show in the Appendix that $\Delta_3 \sim At^{3\alpha}$ [see Eq. (A8) for details], which becomes exact in the asymptotic limit. We can easily check numerically that the coefficient is positive for $\alpha > 1/2$. In the Appendix we also derive for $\alpha > 1/2$ an exact expression for σ which has the form $\sigma = Bt^\alpha$, and therefore

$$\gamma_1 = -\frac{\Delta_3}{\sigma^3} = -\frac{A}{B}, \tag{11}$$

which is not zero. The variation of the skewness as a function of α for $\alpha \geqslant 1/2$ is shown in Fig. 2. The nonzero values of γ_1, even in the long time limit, contradict the assumption of a Gaussian distribution. We have thus proved that the distribution is non-Gaussian in the superdiffusive regime. The Fokker-Planck equation (4) is not able to produce such moments unless it is amended.

The surprised reader may ask, could this skewness be an artifact of the initial bias? The biased initial condition $x_1(1) = X_1 = 1$ is required for the nonzero value we found for Δ_3. A nonbiased initial condition, i.e., $x_1(1) = 0$, leads to $x_1(t) = 0$ [from Eq. (8)] with $n = 3$. Since $x_3(1) = 0$, we get $x_3(t) = 0$ [from Eq. (8)] with $n = 3$. This leads to $\Delta_3 = 0$ and therefore $\gamma_1 = 0$. In fact, the odd moments are all zero if $x_1(1) = 0$, leading to a symmetric distribution. Although this distribution is symmetric, it is not Gaussian. In fact, using (8), one can show that the kurtosis $\gamma_2 = \langle [X_t - x_1(t)]^4 \rangle/\sigma^4 - 3$ is nonzero for the nonbiased initial condition. Therefore the propagator, represented by the distribution of $\{X_t\}$, is indeed non-Gaussian, despite the even symmetry of the distribution for the symmetric initial distribution.

B. The Fokker-Planck equation

Having shown that the propagator is non-Gaussian, we now return to the Fokker-Planck equation. Clearly, the moments of the distribution given by recurrence relation (10) are relative to a non-Gaussian distribution. Moreover, these recurrence relations were derived directly from the definition of the ERW. They are not approximate, but exact [unlike Fokker-Planck equation (4), which neglects higher order terms]. The Gaussian solution to the Fokker-Planck equation (4) is not able to produce such moments, which suggests that this equation should be reviewed.

Indeed, our final goal is to derive a Fokker-Planck equation for the ERW with the next higher order correction terms. For completeness, we first study the equations of motion of the random walker, which can be obtained from Eq. (10) by subtracting $x_n(t)$ from both sides, giving

$$S_n(t) \equiv x_n(t+1) - x_n(t) = \delta + \frac{n\alpha}{t}x_n(t) + \sum_{l=1}^{s(n)}\left[\binom{n}{2l} + \frac{\alpha}{t}\binom{n}{2l+1}\right]x_{n-2l}(t). \tag{12}$$

Expanding this expression in a Taylor series, we finally get the equation of motion:

$$\frac{d}{dt}x_n(t) + \frac{1}{2}\frac{d^2x_n(t)}{dt^2} + \cdots = S_n(t). \tag{13}$$

In order to obtain the correction terms for the Fokker-Planck equation, we start with the discrete (exact) equation (3). After subtracting $P(Y,t|X_0,0)$ from both sides and writing $x = Y$, we can write

$$\begin{aligned} &P(x,t+1) - P(x,t) \\ &= \frac{P(x+1,t) - 2P(x,t) + P(x-1,t)}{2} \\ &\quad - \frac{\alpha}{t}\left[\frac{(x+1)P(x+1,t) - (x-1)P(x-1,t)}{2}\right]. \end{aligned} \tag{14}$$

A Taylor series representation of this equation can be written in closed form as

$$\sum_{i=1}^{\infty}\frac{1}{i!}\frac{\partial^i}{\partial t^i}P(x,t) = \sum_{i=1}^{\infty}\left(\frac{1}{(2i)!}\frac{\partial^{2i}}{\partial x^{2i}}P(x,t) - \frac{\alpha}{t}\frac{1}{(2i-1)!}\frac{\partial^{2i-1}}{\partial x^{2i-1}}[xP(x,t)]\right), \tag{15}$$

which is the complete form of the Fokker-Planck equation for the ERW model. The first few terms can be rearranged to give

$$\frac{\partial}{\partial t}P(x,t)+\frac{1}{2}\frac{\partial^2}{\partial t^2}P(x,t)+\cdots$$
$$=\frac{1}{2}\left(1-\frac{\alpha}{t}\right)\frac{\partial^2}{\partial x^2}P(x,t)-\frac{\alpha}{t}\frac{\partial}{\partial x}[xP(x,t)]$$
$$-\frac{\alpha}{t}\frac{x}{6}\frac{\partial^3}{\partial x^3}P(x,t)+\frac{1}{24}\left(1-\frac{\alpha}{t}\right)\frac{\partial^4}{\partial x^4}P(x,t)+\cdots,$$

which must be compared with Eq. (4). We see now that Eq. (4) can be regarded as a lowest order Fokker-Planck equation for the ERW, valid for large t. This lowest order Fokker-Planck equation leads to the correct scaling exponents for the first and second moments. Even for higher order moments (third and so on), the scaling exponents are given by (4), but the scaling coefficients are likely incorrect. In fact, within the superdiffusive regime, the coefficients will certainly be affected by neglecting the higher order derivatives in the Fokker-Planck equation. This affects mainly the central moments $\langle [X_t - x_1(t)]^n\rangle$.

In view of the generalized form of the Fokker-Planck equation (15), it is not surprising that the Gaussian statistics should not prevail within all diffusion regimes. In fact, for $\alpha > 0$ one can easily show that the lowest order Fokker-Planck equation (4) admits a second, modified Gaussian solution, namely, P^*, in addition to the known Gaussian solution. Notice, however, that the existence of a second solution has no physical meaning. Indeed, Eq. (4) is valid only in lowest order. Direct substitution shows that $P^*(x,t)$ is, in fact, a second solution of the Fokker-Planck equation (4). We next find the function $g(t)$. For large t we can derive analytic expressions for $g(t)$ in the form

$$g(t)=\begin{cases}\beta_s/t^{\alpha} & (\alpha>1/2),\\ A/[t^{1/2}\ln t] & (\alpha=1/2),\\ \beta_e/t^{1-\alpha} & (0<\alpha<1/2),\end{cases} \tag{16}$$

for superdiffusion, marginal diffusion, and normal diffusion with escape, respectively, where s and e stand for superdiffusive and diffusive with escape regimes. For $\alpha < 0$ the functional form of $g(t)$ is similar to that in the interval $0 < \alpha < 1/2$. However, in this case, the function $g(t)(x-\overline{x})$ vanishes altogether for large t, for any x, implying Gaussian statistics, which agrees with the original results in [13]. For both the superdiffusion and escape regimes, one can easily show (see below) that β_s is nonzero and is given by $\beta_s = \beta_e = |(2\alpha-1)(1/\Gamma(\alpha+1)-1)|$. This result is in agreement with the non-Gaussian statistics associated with the superdiffusive regime proved above. Also for $0 < \alpha < 1/2$ this points towards a non-Gaussian statistics, but the skewness and the kurtosis in this region go to zero asymptotically (not shown). On the other hand, we did not find convincing numerical evidence regarding non-Gaussian behavior in this region. Therefore more studies are needed to determine the statistics for $0 < \alpha < 1/2$.

We must remark that $P^*(x,t)$ must be positive in order to represent a physical solution. Therefore $[1 + g(t)(x-\overline{x})]$ must be greater than zero. For $\alpha > 1/2$, this can be achieved by setting $|x-\overline{x}| < 1/g(t)$ or $|x-\overline{x}| < t^{\alpha}/\beta_s$. Therefore the positiveness condition poses a problem in the tail of the distribution, which is hardly present in the simulation. One could argue that the overall shape of the distribution would not be affected by removing such points by setting a cutoff for this task.

V. CONCLUSIONS

In summary, we have investigated the statistical behavior of the ERW. Motivated by numerical evidence of a non-Gaussian propagator, we proved that the propagator is, in fact, non-Gaussian in the superdiffusive regime. In the diffusion regime $p \leqslant 1/2$ the propagator is definitely Gaussian. In the intermediate regime $1/2 < p < 3/4$ the situation is still not completely clear, and the problem remains wide open. A closed form for the Fokker-Planck equation has also been derived with higher order terms included. We strongly believe that these results will motivate further studies in the area in order to establish the full solution of this non-Markovian problem. In particular, future work on the ERW may elucidate the long time influence of the initial steps in non-Markovian random walks. At this very moment, the model is being extended to two dimensions, and it may find suitable applications to, e.g., walks of small insects, driven by internal communications within the colony. The memory retained is an essential feature to avoid random directions. It may also find applications in other fields if one considers, for example, the possibility of introducing memory damage, which causes the advent of log periodicity, a distinguished feature, and ubiquitous characteristics of crash events.

Note added in proof. Recently we became aware that Gaussian deviations in the superdiffusive regime of the ERW have also been pointed out by Paraan and Esguerra [6]. Their work focuses on a continuous-time random walk (CTRW) generalization of the ERW, which is a different approach than the discrete-time calculations described in our paper.

ACKNOWLEDGMENTS

J.C.C. and M.A.A.S. thank FAPESP (Grants No. 2011/13685-6 and No. 2011/06757-0) and G.M.V. thanks CNPq for funding. G.M.S. thanks FAPESP (Grant No. 2011/21089-4) for financial support and C. Coletti for useful discussions.

APPENDIX: PROOF OF THE NON-GAUSSIAN BEHAVIOR IN THE SUPERDIFFUSIVE REGIME

In this section we provide an analytic proof that settles definitely that the position distribution is non-Gaussian for $\alpha > 1/2$ $(p > 3/4)$. This is done by showing that the skewness $\gamma_1 = -\Delta_3/\sigma^3$, with $\Delta_3 = \langle (x_1(t) - X_t)^3\rangle$ and $\sigma = \langle [X_t - x_1(t)]^2\rangle^{1/2}$, is nonzero in the superdiffusive regime.

Starting with the moments, we compare Eqs. (8) and (10). For the first moment we can write

$$x_1(t+1)=\left(1+\frac{\alpha}{t}\right)x_1(t),$$

from which $h_1(t)=0$ and $g_1(t)=1+\alpha/t$ follow. The solution for this equation is simply given by

$$x_1(t)=x_1(1)\prod_{k=1}^{t-1}\left(1+\frac{\alpha}{k}\right),$$

and since $x_1(1)=1$, we get

$$x_1(t)=\prod_{k=1}^{t-1}\left(1+\frac{\alpha}{k}\right)=\frac{\Gamma(t+\alpha)}{\Gamma(t)\Gamma(1+\alpha)}. \tag{A1}$$

For the second moment we can write

$$x_2(t+1)=1+\left(1+\frac{2\alpha}{t}\right)x_2(t),$$

which leads to $h_2(t)=1$ and $g_2(t)=1+2\alpha/t$. Thus we can write

$$x_2(t)=x_2(1)\prod_{k=1}^{t-1}\left(1+\frac{2\alpha}{k}\right)+\sum_{m=1}^{t-1},\prod_{k=m+1}^{t-1}\left(1+\frac{2\alpha}{k}\right)$$

but since $x_2(1)=1$, we get

$$\begin{aligned}x_2(t)&=\prod_{k=1}^{t-1}\left(1+\frac{2\alpha}{k}\right)+\sum_{m=1}^{t-1}\prod_{k=m+1}^{t-1}\left(1+\frac{2\alpha}{k}\right)\\&=\frac{t}{2\alpha-1}\left(\frac{\Gamma(t+2\alpha)}{\Gamma(2\alpha)\Gamma(t+1)}-1\right).\end{aligned} \tag{A2}$$

Finally, for the third moment we can write

$$\begin{aligned}x_3(t+1)&=\left(1+\frac{3\alpha}{t}\right)x_3(t)\\&\quad+\sum_{l=1}^{1}\left[\binom{n}{2l}+\binom{n}{2l+1}\frac{\alpha}{t}\right]x_{3-2l}(t)\\&=\left(1+\frac{3\alpha}{t}\right)x_3(t)+\left(3+\frac{\alpha}{t}\right)x_1(t).\end{aligned}$$

However, since

$$\begin{aligned}h_3(t)&=\left(3+\frac{\alpha}{t}\right)x_1(t)\\&=\left(3+\frac{\alpha}{t}\right)\frac{\Gamma(t+\alpha)}{\Gamma(t)\Gamma(1+\alpha)}\end{aligned}$$

and

$$g_3(t)=1+\frac{3\alpha}{t},$$

the solution [with $x_3(1)=1$] becomes

$$\begin{aligned}x_3(t)&=x_3(1)\prod_{k=1}^{t-1}g_3(k)+\sum_{m=1}^{t-1}\left[h_3(m)\prod_{k=m+1}^{t-1}g_3(k)\right]\\&=\prod_{k=1}^{t-1}\left(1+\frac{3\alpha}{k}\right)+\sum_{m=1}^{t-1}\left[\left(3+\frac{\alpha}{m}\right)\frac{\Gamma(m+\alpha)}{\Gamma(m)\Gamma(1+\alpha)}\right.\\&\quad\left.\times\prod_{k=m+1}^{t-1}\left(1+\frac{3\alpha}{k}\right)\right],\end{aligned}$$

which can be written as

$$\begin{aligned}x_3(t)&=\frac{1+\alpha}{(2\alpha-1)\Gamma(3\alpha)\alpha}\frac{\Gamma(t+3\alpha)}{\Gamma(t)}\\&\quad-\frac{1+\alpha+3t}{(2\alpha-1)\alpha\Gamma(\alpha)}\frac{\Gamma(t+\alpha)}{\Gamma(t)}.\end{aligned} \tag{A3}$$

We can now evaluate Δ_3, defined by

$$\begin{aligned}\Delta_3&\equiv\langle[x_1(t)]^3-3[x_1(t)]^2x+3x_1(t)x^2-x^3\rangle\\&=-2[x_1(t)]^3+3x_1(t)x_2(t)-x_3(t).\end{aligned} \tag{A4}$$

Using equations (A1), (A2), and (A3), we can write

$$\begin{aligned}\Delta_3&=-\frac{1+\alpha}{(2\alpha-1)\Gamma(3\alpha)\alpha}\frac{\Gamma(t+3\alpha)}{\Gamma(t)}\\&\quad+\frac{1+\alpha}{(2\alpha-1)\alpha\Gamma(\alpha)}\frac{\Gamma(t+\alpha)}{\Gamma(t)}\\&\quad+\frac{3}{\alpha(2\alpha-1)\Gamma(\alpha)\Gamma(2\alpha)}\frac{\Gamma(t+\alpha)}{\Gamma(t)}\frac{\Gamma(t+2\alpha)}{\Gamma(t)}\\&\quad-\frac{2}{\Gamma^3(\alpha)\alpha^3}\left(\frac{\Gamma(t+\alpha)}{\Gamma(t)}\right)^3,\end{aligned} \tag{A5}$$

valid for all α. Using the asymptotic expansion for $\Gamma(t)$, i.e.,

$$\Gamma(t)\sim t^{t-1/2}e^{-t}\sqrt{2\pi}\left(1+\frac{1}{12t}+\frac{1}{288t^2}-\cdots\right),$$

and keeping only terms up to the order t^{-1}, we have

$$\begin{aligned}\frac{\Gamma(t+\alpha)}{\Gamma(t)}&\sim\frac{(t+\alpha)^{t+\alpha-1/2}e^{-(t+\alpha)}\sqrt{2\pi}\left(1+\frac{1}{12(t+\alpha)}\right)}{t^{t-1/2}e^{-t}\sqrt{2\pi}\left(1+\frac{1}{12t}\right)}\\&=\frac{(t+\alpha)^{t+\alpha-3/2}}{t^{t-3/2}}\left(\frac{12t+12\alpha+1}{12t+1}\right)e^{-\alpha}\\&\sim\left(1+\frac{1}{2}\alpha(2\alpha-1)\frac{1}{t}\right)t^{\alpha}.\end{aligned} \tag{A6}$$

Therefore we can write

$$\frac{\Gamma(t+2\alpha)}{\Gamma(t)}\sim\left(1+\alpha(4\alpha-1)\frac{1}{t}\right)t^{2\alpha}$$

and

$$\frac{\Gamma(t+3\alpha)}{\Gamma(t)}\sim\left(1+\frac{3}{2}\alpha(6\alpha-1)\frac{1}{t}\right)t^{3\alpha}$$

and, finally,

$$\left(\frac{\Gamma(t+\alpha)}{\Gamma(t)}\right)^3\sim\left(1+\frac{3}{2}\alpha(2\alpha-1)\frac{1}{t}\right)t^{3\alpha}. \tag{A7}$$

Inserting these back into Eq. (A5) and keeping only the dominant terms, we get

$$\begin{aligned}\Delta_3&\sim\left(\frac{3}{\alpha(2\alpha-1)\Gamma(\alpha)\Gamma(2\alpha)}\right.\\&\quad\left.-\frac{1+\alpha}{\alpha(2\alpha-1)\Gamma(3\alpha)}-\frac{2}{\Gamma^3(\alpha)\alpha^3}\right)t^{3\alpha}=At^{3\alpha},\end{aligned} \tag{A8}$$

which becomes exact in the asymptotic limit. We can easily check numerically that the coefficient is positive for $\alpha>0$.

On the other hand, the standard deviation $\sigma=[x_2(t)-x_1^2(t)]^{1/2}$ can be derived using expressions (A1) and (A2). For

$\alpha < 1/2$ this leads to $\sigma \sim t$. For $\alpha > 1/2$ it can be shown that

$$\sigma \sim \left(\frac{1}{(2\alpha - 1)\Gamma(2\alpha)} - \frac{1}{\Gamma^2(1+\alpha)} \right)^{1/2} t^\alpha \sim B t^\alpha, \quad \text{(A9)}$$

and therefore the skewness γ_1 for $\alpha > 1/2$ can be written as

$$\gamma_1 = -\frac{\Delta_3}{\sigma^3} = -\frac{A}{B}. \quad \text{(A10)}$$

We included a negative sign due to the definition of Δ_3 in Eq. (A4). Figure 2 shows the variation of the skewness as a function of α for $\alpha > 1/2$. The nonzero values of γ_1 imply that the distribution is non-Gaussian in the superdiffusive regime. We conclude that the moments of the distribution given by the recurrence relation in Eq. (10) are relative to a non-Gaussian distribution.

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